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Schemes of Quantum Computations with Trapped Ions

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Abstract

The purpose of this article is to review two complementary schemes for quantum computation with trapped ions. We initially discuss the first proposal of quantum computations with cold trapped ions (J. I. CIRAC and P. ZOLLER; Phys. Rev. Lett. **74**, 4091, 1995) which requires, prior to any computation, laser cooling to the motional ground state. This proposal is closely related to the physics of generating and manipulating N-particles entangled states in both ion traps and high-Q cavities (cavity quantum electrodynamics). The second scheme is that of quantum computations with *hot* trapped ions (J. F. POYATOS, J. I. CIRAC and P. ZOLLER; Phys. Rev. Lett. **81**, 1322, 1998) which works at finite temperature and resembles physical ideas found in atom interferometry.

1. Introduction

Initial discussions about the ultimate limits in size of computers seemed to be very far away from any realistic computing device not so long ago. Even within this purely academic domain, the subject attracted great interest once the arguments related to reversible computation led unavoidably into quantum physics [1]. In particular, recent results in quantum complexity and algorithmic theory indicated that quantum computers would solve problems efficiently which are considered intractable on classical Turing machines [2]. Another reason for the interest in quantum devices is that the energy consumption of a computation can be minimized. This is relevant since heat production is one of the limiting factors in present computing technology [1]. All these reasons triggered the interest in finding experimental setups to implement such quantum computing devices. Although it is still not clear whether a fully fledged quantum computer will be ever built in the future, small–scale devices are being constructed in different laboratories worldwide. With these small–scale processors it is already possible to generate deterministically quantum states with no classical analogy (allowing possible relevant applications in the improvement of time and frequency standards) [3]. These devices will also help the experimental study of topics such decoherence, quantum measurement and other fundamental concepts of quantum mechanics.

The aim of this article is to survey in a detailed way two complementary proposals related to quantum computations in the experimental context of ion traps [4, 5]. The paper is organized as follows. Firstly, we introduce the ion trap setup, basic tools, and their connection with basic computational ideas. We discuss then the cold and hot schemes, presenting in detail the models and interactions required to implement quantum computations. After this, we briefly discuss issues of decoherence and error correction in such devices. We end up with a comparison between the two schemes.

2. The Ion Trap Setup

One of the initial advantages of the use of trapped ions was that many of the requirements needed for implementing quantum computations were already advanced in relation to their initial application in high precision spectroscopy and frequency standards. In this section we introduce some of this physics, which will become basic tools in the following discussions [6].

- *Ion trapping.* Nowadays it is possible to trap and confine atomic particles for a long time in a small region in space. In the case of ions one can make use of cleverly designed electric and magnetic fields to create such traps. In the case of a single ion Paul trap a combination between static and oscillating potentials makes possible to create an harmonic 'pseudopotential'. It is possible then to trap just one single ion in the place where the potential is minimum.
  For quantum computing a linear radio frequency Paul trap is preferably used. The device consists of four parallel rods. A voltage varying at radio frequency is applied to two opposing rods while the other two are grounded. In the case of the single Paul trap it is possible to trap just one single ion in the place where the potential is minimum, due to the Coulomb repulsion. On the other hand, in the case of the linear Paul trap it is possible to get a zero radio frequency field not just in a single point but in a line, allowing the trapping of several ions without any residual heating.
- *Laser cooling.* This consists essentially in slowing down rapidly ion motion by means of its interaction with laser light. This can effectively decrease the temperature of the ions from room temperature down to less than 1 K. The basic idea relies on the practical use of radiation pressure, the mechanical force associated to every laser beam. Such radiation pressure, small in a macroscopic scale, is big enough in a microscopic scale (of the order of 10.000 times that due to gravity). To slow down the atoms, laser beams in opposite directions are applied. By means of the Doppler effect only those ions moving toward the laser propagation will absorb light and thus decrease their momentum.
- *Laser manipulation.* Rabi floppings, i.e. coherent transitions between the internal states, are performed by applying controlled laser pulses. The particular ion–laser interaction will be fundamentally different depending on the intensity of the laser. We will discuss in detail such regimes in the following sections. An important parameter associated to laser cooling and laser manipulation is the Lamb–Dicke parameter, which relates the characteristic length scale of motion of the ion and the wavelength of the exciting laser light. In the so–called Lamb-Dicke limit the wavelength of the light is larger than the ion characteristic length scale.
- *The quantum jump technique.* Measurements of the internal quantum state of the ions are performed using the so-called quantum jump technique. It is possible to measure the state of the internal level by considering a three–level V configuration where one transition is stronger than the other, see Fig. 1. A laser beam is tuned to the strong transition to measure the state of the two–level system associated with the weak one. This is measured by the presence (absence) of spontaneously emitted photons from the strong one.

3. Quantum Computation: Basic Elements in an Ion Trap

The task of designing a quantum computer is equivalent to finding a physical realization of quantum gates between a set of qubits. We get now more explicit about this, enumerating in detail basic computational elements and the way they can be implemented with trapped ions.

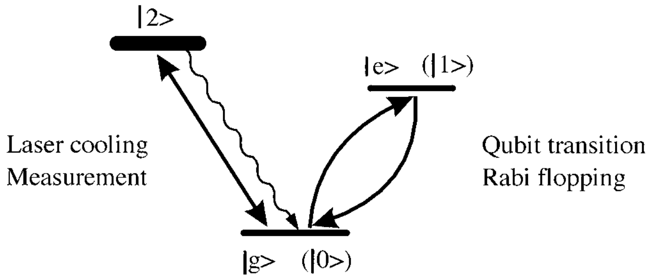

Fig. 1: Laser cooling, coherent manipulation and state measurement is possible by using the internal (electronic) structure of the ions. The internal levels associated to the weak transitions play the role of the qubits, $(|0\rangle, |1\rangle)$. Transition $|2\rangle \leftrightarrow |0\rangle$ is a dipole allowed transition used for cooling and detection, by means of the quantum jumps technique.

– *Qubits:* The basic elements of the computer. In this case the qubits are the trapped ions themselves. We need the ion to be in one of two states which will be identified with the internal electronic states; for example a ground state $|g\rangle$ plays the role of the (quantum) zero state $|0\rangle$ and a excited state $|e\rangle$ plays the role of one, $|1\rangle$, see Fig. 1. The classical case would correspond to the situation where the qubit is prepared in one of these states. However, quantum mechanically we can have any arbitrary superposition of the two basic states. This requires the system to be very well isolated from its environment. In the ion trap setup arbitrary coherent superpositions of internal states can be stored for a long time, longer that the time of any typical computation. This is because very long–lived electronic levels are used to store the qubits. Finally, a string of ions represents a quantum register.
– *Gates:* It is sufficient to find how to implement an adequate set of logic gates, in particular single and two–qubit gates, to perform any general computation [7]. In the next section we describe in detail how to realize quantum gates in two different regimes, associated to two different temperatures. Also note that 2-bit quantum gates will be harder to implement since they require a physical mechanism to strongly couple any pair of qubits.
– *Input/Output:* In order to realize any computation we need to both initialize the state of the computer and to read out the final state of the computation. Initialization is possible by using laser cooling techniques. It is possible to optically pump the ion to the state where all qubits are $|0\rangle$. Once the computation is finished, the readout, i.e. the state

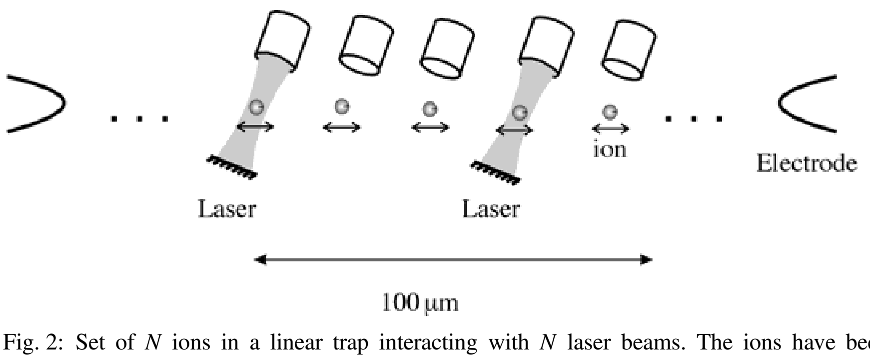

Fig. 2: Set of N ions in a linear trap interacting with N laser beams. The ions have been previously laser cooled to the ground state of motion. The typical length scale of the quantum register is 100 μm.

measurement, of the quantum register is possible using the quantum jump technique previously discussed. Readout can play a role in error corrections schemes likewise. Finally, note that measurements require a strong coupling of the quantum system to the environment which is in strong contrast to the initially discussed isolation requirement. It follows that we need a means to switch on and off the interaction of any arbitrary qubit to the environment deliberately.

4. Cold Gates

In this section we describe the proposal for quantum computations with *cold* trapped ions [4]. In this scheme, illustrated in Fig. 2, each qubit has been laser cooled to its ground state of motion in a linear Paul trap. Computer operations are realized by means of coherent laser–ion interactions inducing Rabi oscillations between the selected quantum states of the qubit register. Readout at the end of the computation is done using the quantum jumps technique.

4.1. Model

Imagine the situation where a string of N ions have been stored in a linear Paul trap. Moreover the trapping potential is designed such that the frequencies characterizing the motion in the x, y direction are much larger than the one in the z direction. We can simplify the description accordingly by considering the motion unidimensional along this last direction. We consider the trapping potential in this direction in the most general way, i.e. not necessarily harmonic but given by $V_{\rm tr}^i = \xi z_i^\alpha$. Motion is also influenced by the Coulomb repulsion between ions. The Hamiltonian of the system is given by $H_{\rm ex} = T + V$ where

$$T = \sum_{i=1}^{N} \frac{P_i^2}{2m} ,$$
$$V = \sum_{i=1}^{N} \left(V_{\rm tr}^i + \sum_{i>j} \frac{Q^2}{|z_i - z_j|} \right) . \quad (1)$$

This Hamiltonian can be simplified as follows. Since the ions have been previously laser cooled in all three dimensions (as a requirement, it is necessary to fulfill the Lamb-Dicke limit for all the modes), the ions undergo very small oscillations around the equilibrium position. We can expand the potential in a Taylor series around the equilibrium position

$$V(r) = \sum_{i=1}^{N} \sum_{j=1}^{N} s_i A_{ij} s_j + O(s_{i,j}^3) , \quad (2)$$

where

$$A_{ii} = \frac{1}{2}\, \alpha(\alpha - 1)\, (r_i^{\rm eq})^{\alpha-2} + \sum_{j \neq i}^{N} \frac{1}{|r_i^{\rm eq} - r_j^{\rm eq}|^3} ,$$
$$A_{ij} = -\frac{1}{|r_i^{\rm eq} - r_j^{\rm eq}|^3} \quad (i \neq j) . \quad (3)$$

Here $s_i \equiv r_i - r_i^{\text{eq}}$, r is an arbitrary displacement direction, z in our case, and we used the following units: $m = Q = \xi = 1$. The motion of the ions can be expressed in terms of normal modes of oscillation. We reexpress the Hamiltonian in terms of the eigenvectors of the matrix A, where $D = \text{diag}\,(\lambda_1, \lambda_2, \ldots)$ and $\mathcal{U}$ are respectively the eigenvalue and eigenvector matrices. The problem is thus equivalent to a set of N uncoupled oscillators. We finally obtain

$$H_{\text{ex}} = \sum_{i=1}^{N} \left(\frac{q_i^2}{2m} + \frac{1}{2}\, m\nu_i^2 z_i^2 \right) = \sum_{i=1}^{N} \hbar\nu_i a_i^\dagger a_i\,, \tag{4}$$

where we have quantized the harmonic oscillators describing the independent normal modes as usual. For the case where the confinement potential is harmonic we have $\nu_i = \nu_z, \sqrt{3}\,\nu_z, \sqrt{5.84}, \nu_z \ldots$. Here the first mode of motion corresponds to the center of mass (CM) motion along the z direction.

4.2. Laser–ion interaction

Let us consider now the interaction of a particular ion i with a standing laser wave (a travelling wave can be treated in the same way). The Hamiltonian describing this situation is given, in a frame rotating with the laser frequency, by $H = H_{\text{ex}} + H_{\text{int}} + H_{\text{las}}$ where ($\hbar = 1$)

$$\begin{aligned} H_{\text{int}}^i &= -\frac{\delta_i}{2}\,\sigma_z^i\,, \\ H_{\text{las}}^i &= \frac{\Omega_i}{2}\,\sin\,(kr_i + \varphi_i)\,(\sigma_i^+ + \sigma_i^-)\,. \end{aligned} \tag{5}$$

Here $\delta_i = \omega_{\text{L}}^i - \omega_0^i$ is the laser detuning, Ω_i the Rabi frequency, k the laser wave vector, φ_i the phase describing the ion–laser standing wave relative position and r_i the ion position, expressed as a combination of normal modes as $r_i = r_i^{\text{eq}} + \sum_{j=1}^N \sqrt{\frac{1}{2m\nu_j}}\, \mathcal{U}_{ij}(a_j + a_j^\dagger)$. We have also considered the Pauli spin operators associated to two level systems and the creation and annihilation operators, associated to the harmonic oscillator. Note also that the laser beam will be applied normally in a direction forming a given angle to the z axis and thus k will be given by $k_\theta = k\cos\,(\theta)$.

Interaction with the laser light will induce transitions between the (internal) ground and excited levels of the ion. These transitions will be able to change the motion of the ions as well, modifying the states of the normal modes. The situation is different in the Lamb–Dicke limit and for weak intensities. In this case laser induced transitions only change the state of one of the modes. Thus, the description of the dynamics is simplified as follows

- Lamb-Dicke limit: Within this limit it is possible to expand the interaction Hamiltonian in terms of the Lamb-Dicke parameter $\left(\eta_j = k\sqrt{\frac{1}{2m\nu_j}} \ll 1\right)$, keeping only zero order terms (in the case when the equilibrium position of the ion coincides with the antinode of the standing wave) or first order terms (in the case when it coincides with the node of the laser standing wave).
- Weak excitation limit: For weak enough laser intensities it will be possible to excite selectively one of the motional modes of the ion.

We can now approximate the ion–laser interaction in the form $H^i_{\rm las} \approx H^i_a + H^i_b$, where

$$H^i_a = \frac{\Omega^a_i}{2} \left(\sigma^+_i\, e^{-i\phi} + \sigma^-_i\, e^{i\phi}\right), \qquad (6)$$
$$H^i_b = \frac{\Omega^b_i}{2}\, \eta_1 \mathcal{U}_{i1} \left(a_1 \sigma^+_i\, e^{-i\phi} + a^\dagger_1 \sigma^-_i\, e^{i\phi}\right).$$

Here η_1 is the Lamb–Dicke parameter associated to ν_1, and ϕ is the laser phase. In the harmonic case $\nu_1 = \nu_z$, $\mathcal{U}_{i1} = 1/\sqrt{N}$, with N the number of trapped ions, and $\eta_1 = \eta = \sqrt{\frac{1}{2m\nu_z}}$. This is the case considered from now on. Note that this formula is valid under the following considerations:

- Either $\Omega^a_1 \neq 0$ ($\delta_a = 0$) or $\Omega^b_1 \neq 0$ ($\delta_b \approx -\nu_1$). That is, there exists two types of transitions modifying, (*b*), or not, (*a*), the motion of the ions.
- Location at the node of the standing wave. In this case the weak excitation limit implies $|\delta + \nu_1|, \eta_1\, \frac{\Omega}{2} \ll \nu_1$.
- Location at the antinode of the standing wave. In this case the weak excitation limit implies $|\delta|, \frac{\Omega}{2} \ll \nu_1$.

4.3. Logic gates

In this section we show how to implement quantum logic gates. Logic gates with just one qubit will be easier to implement, since they imply individual rotations of each ion, without changing the motional state. They can be implemented using a laser on resonance with the internal transition ($\delta_i = 0$). The equilibrium position of the ion coincides with the antinode of the laser standing wave. We have seen in the previous discussion how the evolution in this case will be given by the Hamiltonian H^i_a, inducing the following rotation

$$|g\rangle_i \to \cos\,(k\pi/2)\; |g\rangle_i - i\, e^{i\phi} \sin\,(k\pi/2)\; |e\rangle_i\,,$$
$$|e\rangle_i \to \cos\,(k\pi/2)\; |e\rangle_i - i\, e^{-i\phi} \sin\,(k\pi/2)\; |g\rangle_i\,.$$

We describe now the way to implement quantum gates between two qubits. In this case the frequency of the laser is chosen such that $\delta_i = -\nu_z$, that is only the CM mode is excited, and the ion equilibrium position coincides with the node of the laser standing wave. The laser interaction is given by the previously defined H^i_b. Applying the laser beam for a given time $t = k\pi/(\Omega^b_i \eta_z/\sqrt{N})$ (a $k\pi$ pulse), states will evolve in the following way

$$|g\rangle_i\, |1\rangle \to \cos\,(k\pi/2)\; |g\rangle_i\, |1\rangle - i\, e^{i\phi} \sin\,(k\pi/2)\; |e'\rangle_i\, |0\rangle\,,$$
$$|e'\rangle_i\, |0\rangle \to \cos\,(k\pi/2)\; |e'\rangle_i\, |0\rangle - i\, e^{-i\phi} \sin\,(k\pi/2)\; |g\rangle_i\, |1\rangle\,,$$
$$|g\rangle\, |0\rangle \to |g\rangle\, |0\rangle\,, \qquad (7)$$

where $|0\rangle$ ($|1\rangle$) denotes the state of the center of mass with zero(one) phonon, ϕ is the laser phase and $|e'\rangle$ denotes either the state $|1\rangle$ of the qubit considered or an auxiliar state selectively excited. This selective excitation can be realized by means of different polarizations or laser frequencies, experimentally frequencies are better controlled than polarizations.

The logic gate between two ions *(qubits)* will be implemented as follows (i) By means of a π pulse focused on the first ion we transfer the internal state of this ion to the CM phonon mode, cooled to its motional ground state prior to the execution of the gate. If the first ion is in a excited state a vibration in the string of ions is introduced. On the other hand, if the ion is in the ground state the ions remain in place. This is done in order to convey quantum information between the ions, using the CM mode as a bus. Since all ions participate in the CM motion they all can see the quantum bus. (ii) we introduced a conditional change of sign by means of a 2π pulse focused on the second ion, by using an auxiliary state $|e'\rangle$. (iii) A new π pulse reverses the first step in order to restore the qubit from the CM mode to the internal state of the first ion. This step completes the following quantum gate

$$\begin{array}{ccccccc} & \textbf{(i)} & & \textbf{(ii)} & & \textbf{(iii)} & \\ |g\rangle_1\,|g\rangle_2\,|0\rangle & \longrightarrow & |g\rangle_1\,|g\rangle_2\,|0\rangle & \longrightarrow & |g\rangle_1\,|g\rangle_2\,|0\rangle & \longrightarrow & |g\rangle_1\,|g\rangle_2\,|0\rangle\,, \\ |g\rangle_1\,|e\rangle_2\,|0\rangle & \longrightarrow & |g\rangle_1\,|e\rangle_2\,|0\rangle & \longrightarrow & |g\rangle_1\,|e\rangle_2\,|0\rangle & \longrightarrow & |g\rangle_1\,|e\rangle_2\,|0\rangle\,, \\ |e\rangle_1\,|g\rangle_2\,|0\rangle & \longrightarrow & -i|g\rangle_1\,|g\rangle_2\,|1\rangle & \longrightarrow & i|g\rangle_1\,|g\rangle_2\,|1\rangle & \longrightarrow & |e\rangle_1\,|g\rangle_2\,|0\rangle\,, \\ |e\rangle_1\,|e_0\rangle_2\,|0\rangle & \longrightarrow & -i|g\rangle_1\,|e\rangle_2\,|1\rangle & \longrightarrow & -i|g\rangle_1\,|e\rangle_2\,|1\rangle & \longrightarrow & -|e\rangle_1\,|e\rangle_2\,|0\rangle\,. \end{array} \quad (8)$$

In this way, the effect of this interaction is to change the sign of the states only when both states are in the excited state while the CM mode comes back to the fundamental state $|0\rangle$. Note that this procedure can be generalized to n ions and similar n–bits gates.

5. Hot Gates

It is clear in the previous section that neither internal decoherence nor single qubit operations pose major restrictions to the experimental realization of the computation. Remember that qubits are stored in long–lived internal atomic ground states and that single bit operations are accomplished by directing laser beams to each of the ions. On the other hand, laser cooling to the ground state of motion is still a demanding task [8]. Incomplete cooling of the collective modes to the ground state implies that the phonons introduced during the application of the logic gate would be indistinguishable from those at finite temperature, leading to errors even when $\bar{n} \approx 1$, where $\bar{n}$ refers to the average phonon number of the CM mode. The possibility of relaxing this requirement would be very attractive.

In the following we discuss an alternative proposal to realize a fundamental two-qubit logic gate between two ions in a linear trap at finite temperature [5]. Firstly we introduce the model and the laser-interaction regime considered in this scheme.

5.1. The model

Consider the case of two ions confined in a linear trap. We assume again that the motion is unidimensional in the z direction (recall that this approximation will be valid in the case of a linear trap, where normally $\nu_{x,y} \gg \nu_z$). The Hamiltonian describing this situation will be the one corresponding to a system of two point charged particles in a external potential

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(z_1) + V(z_2) + \frac{Q^2}{|z_1 - z_2|}\,, \quad (9)$$

where V denotes the confinement potential along the z axis, that we assume symmetric $V(z) = V(-z)$. Making use of the relative and center of mass coordinates, and introducing

z_e as the separation of the ions in the equilibrium position,

$$z_c = (z_1 + z_2)/2\,, \qquad z_r = z_2 - z_1 - z_e\,, \tag{10}$$

$$p_c = p_1 + p_2\,, \qquad p_r = (p_2 - p_1)/2\,, \tag{11}$$

we can expand around the equilibrium position $z_c = z_r = 0$. Up to second order, we get

$$H = H_{\rm ho} + V_{\rm cor}(z_c, z_r)\,, \tag{12}$$

where

$$H_{\rm ho} = \frac{p_c^2}{2m_c} + \frac{p_r^2}{2m_r} + \frac{1}{2}\, m_c \nu_c^2 z_c^2 + \frac{1}{2}\, m_r \nu_r^2 z_r^2\,, \tag{13}$$

is the harmonic oscillator Hamiltonian and $V_{\rm cor}$ indicates corrections of third order and beyond. Here $m_c = 2m$ and $m_r = m/2$ are respectively the center of mass and reduced mass of the ions, and $\nu_{c,r}$ the corresponding frequencies in the harmonic approximation

$$\nu_c^2 = \frac{1}{m_c} \left.\frac{\partial^2 V}{\partial z_c^2}\right|_{eq}, \qquad \nu_r^2 = \nu_c^2 + \frac{4Q^2}{m z_e^3}\,. \tag{14}$$

In the following we assume that the trapping potential is such that the frequencies ν_r, ν_c are commensurable, i.e. motion is strictly periodic with period t_g. In practice, this could be realized adding external static fields in such a way that the potential $V(z)$ becomes unharmonic. It will be possible to change the fields in a continuous way until finding the point where the frequencies are commensurable, something which can be easily measured. On the other hand, to be able to use the harmonic approximation related to the original Hamiltonian we will not consider corrections due to the $V_{\rm cor}$ term. This requires $\overline{V_{\rm cor}} t_g \ll 1$, where $\overline{V_{\rm cor}}$ refers to the typical values of the corrections and t_g is the duration of the logic gate. Under this circumstances, the evolution will be given by

$$e^{-iH_{\rm ho} t_g} = 1\,. \tag{15}$$

Note that alternatively one could wait for recurrence times in harmonic traps. We have checked that is better to have commensurable frequencies $\nu_r/\nu_c = 1.5, 2$, etc. For instance, $\nu_r/\nu_c = 1.5$ requires two rings of radius $R = 2$ mm separated by a distance $L \simeq 3$ mm in the configuration of H. C. NÄGERL et al. [Appl. Phys. B **66**, 603 (1998)]. In what follows we assume for simplicity that $\nu_r = 2\nu_c$.

5.2. Laser–ion interaction

In this section, we analyze the interaction of a trapped ion with a laser beam in the so–called strong excitation regime [9]. In this regime, the Rabi frequency associated to the interaction is much bigger than the trap frequency ($\Omega \gg \nu_z$). This means that we can consider the ion at rest when interacting with the laser beam. We also assume no spontaneous emission by tuning the laser to an electric dipole forbidden transition $|g\rangle \leftrightarrow |e\rangle$. Under such conditions, the problem can be exactly solved.

To study this regime let us consider the case of a single ion trapped in an tridimensional harmonic potential. The ion is interacting with a plane laser wave along the z axis. The problem can be reduced to one dimension in this direction since only stimulated transitions are considered. The Hamiltonian describing this situation in a frame rotating with the laser frequency, ω_L, is given by ($\hbar = 1$),

$$H_\pm = \frac{P_z^2}{2m} + \frac{1}{2}\, m\nu_z^2 Z^2 - \frac{\delta}{2}\,\sigma_z + \frac{\Omega}{2}(\sigma_+\, e^{\pm ik_L Z} + \sigma_-\, e^{\mp ik_L Z})\,. \tag{16}$$

Here, Z and P_z are the position and momentum operators, ν_z is the trap frequency in this direction and m is the mass. On the other hand σ_+, σ_- and σ_z are the usual spin $\frac{1}{2}$ operators describing the internal transitions between the levels $|g\rangle \leftrightarrow |e\rangle$, $\delta = \omega_L - \omega_0$ is the laser detuning, $k_L = \omega_L/c$ the associated wavevector, and Ω the Rabi frequency. Subindex "+" ("−") indicates that the plane wave is propagating toward the positive (negative) values of x.

The evolution given by (16) can be easily understood if we realize the following unitary operation $U_\pm = e^{\mp ik_L Z\,|e\rangle\,\langle e|}$. Using this operator, states are defined as $|\tilde{\Psi}_\pm\rangle = U_\pm\,|\Psi\rangle$, while the Hamiltonian can be rewritten as

$$\tilde{H}_\pm \equiv U_\pm H_\pm U_\pm^\dagger = \nu_z a^\dagger a \pm i\nu_z\eta_z(a - a^\dagger)\,|e\rangle\,\langle e| + \nu_z\eta_z^2\,|e\rangle\,\langle e| - \frac{\delta}{2}\,\sigma_z + \frac{\Omega}{2}\,\sigma_x\,, \tag{17}$$

where $\sigma_x = \sigma_+ + \sigma_-$, and we have expressed the position and momentum operators as a function of the creation, annihilation operators of the harmonic oscillators, a and $a^\dagger$, respectively, and the Lamb–Dicke parameter $\eta_z = k_L/\sqrt{2m\nu_z}$. The new terms in the Hamiltonian (17) correspond to the Doppler recoil energy of the excited state (that is, when one photon is absorbed).

Hamiltonian (17) can be simplified in the strong excitation regime, $\Omega \gg \nu_z$. Consider a laser pulse whose duration τ satisfies $\nu_z\tau\,\max(\bar{n}, \eta_z^2) \ll 1$, where $\bar{n} = \langle a^\dagger a\rangle$ (this is too restrictive a condition; see [9] for details). Under this condition, the first terms in the Hamiltonian (17) will not affect practically the evolution. In this case, supposing for simplicity zero detuning ($\delta = 0$), the Hamiltonian reduces to $\tilde{H} = \frac{1}{2}\Omega\sigma_x$ (for both cases, $\tilde{H}_\pm$). The evolution for any initial state can be derived, obtaining $|\tilde{\Psi}_\pm(\tau)\rangle = U_\pm^\dagger\, e^{-i\frac{1}{2}\Omega\sigma_x\tau}\, U_\pm\,|\Psi(0)\rangle$. It is easy to see that the evolution given by H_- can be obtained from that given by H_+ by simply substituting $\eta_z \to -\eta_z$. In particular for an initial coherent state $|\alpha\rangle$ we obtain

$$\begin{aligned} &|g\rangle\,|\alpha\rangle \to A\,|g\rangle\,|\alpha\rangle + B\,|e\rangle\,|\alpha \pm i\eta_z\rangle\,, \\ &|e\rangle\,|\alpha\rangle \to A^*\,|e\rangle\,|\alpha\rangle - B^*\,|g\rangle\,|\alpha \mp i\eta_z\rangle\,, \end{aligned} \tag{18}$$

where $A = \cos\,(\Omega\tau/2)$, $B = -i\sin\,(\Omega\tau/2)$ and the plus (minus) sign corresponds to a laser propagating toward the positive (negative) values of the z axis. Physically this interaction describes the flip of the internal state of the ion and the displacement associated to the absorption/emittion of a stimulated photon.

5.3. Logic gate

Consider the previous situation of two ions confined in a linear trap. The present proposal is based on the following physical concept: according to Fig. 3 we split the wave packet of the so-called *control* ion (ion 1) into two (ϕ_1^R and ϕ_1^L) moving into different directions (right and left) depending on its internal state. Coulomb interaction will divide

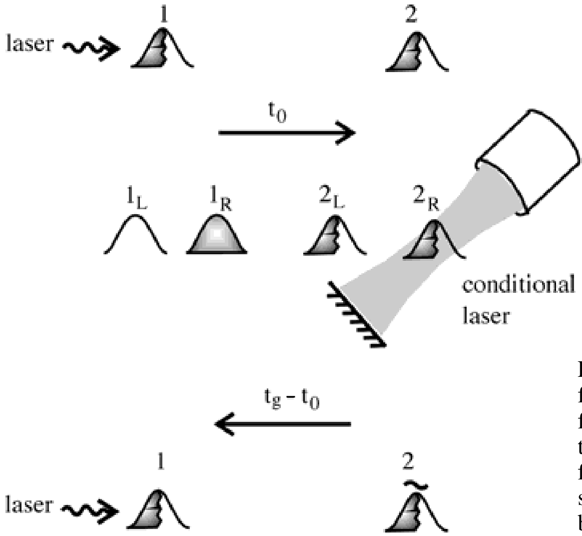

Fig. 3: 2-qubit gate based on atom interferometry. The atomic wave packet of the first ion is split using laser light. After a time t_0 a laser is applied to the second, ion flipping its internal state conditional on the state of the first. Finally another laser brings back the ion to the initial condition.

at the same time the so–called target ion (ion 2) into two (ϕ_2^R and ϕ_2^L) respectively. After a time t_0, a laser will be focused to the center of the wave packet ϕ_2^R, inducing a change in its internal state ($|0\rangle \leftrightarrow |1\rangle$). Finally after a time $t_g - t_0$ the first ion is brought back to its original state by means of another laser, giving as a result the following fundamental two qubit gate

$$\hat{C}_{12} : |\epsilon_1\rangle\, |\epsilon_2\rangle \rightarrow |\epsilon_1\rangle\, |(1-\epsilon_1) \oplus \epsilon_2\rangle \qquad (\epsilon_{1,2} = 0, 1)\,, \tag{19}$$

where $\oplus$ denotes addition mod two.

The new concept used in the implementation of the gate is the splitting of the atomic wave packets in the same way that is done in atom interferometry. After this, it is possible to realize conditional dynamics associated to the spatial localization of the wave packets. Under ideal conditions, the realization of the logic gate is independent of the state of motion of the ions without any restriction of zero temperature. On the other hand the scheme requires the motion of the ions to be periodic. This requirement will imply the need for a nonharmonic trap, since for a external harmonic potential the frequencies of the collective modes are typically inconmensurable (for the case of two ions, the ratio between the frequencies of the relative and the center of mass mode is $\sqrt{3}$). We highlight again that this proposal is fundamentally different to that formerly discussed, where the logic gates between ions were implemented by means of a selective excitation of the center of mass mode which had been previously cooled to the ground state.

We extend now in more detail the scheme just introduced. We denote by $|0\rangle$ and $|1\rangle$ the internal states of the ions storing the quantum information and by $|\Psi(0)\rangle = |\Psi_{\rm int}\rangle \otimes |\Psi_{\rm mov}\rangle$ the initial state of them. Note that we have considered here pure states but our analysis will be equally valid for mixtures. The logic 2-bit quantum gate will be realized in three steps

(i) Displacement of ion 1, depending on its internal state: a laser pulse is applied to ion 1 with the wavevector k pointing along z axis (this can be realized using two Raman beams such that the difference between their wavevectors k points along z). The laser intensity is chosen in such a way that the internal state of the ions is flipped $|0\rangle \leftrightarrow |1\rangle$. The motion of the ion is also modified by this laser beam. We write the interaction Hamiltonian between laser and ion as

$$H_{\text{las}} = \frac{\Omega}{2}\,(\sigma_1^+\,e^{ikz_1} + \sigma_1^-\,e^{-ikz_1})\,, \tag{20}$$

where Ω is the Rabi frequency, $\sigma^+ = |1\rangle\,\langle 0| = (\sigma^-)^\dagger$ and the subindex indicates from now on which of the ions comes into play. The laser is applied for a time $t_{\text{las}} = \pi/\Omega \ll t_g$ (this is the previously discussed strong excitation regime since $t_g \sim \nu_z^{-1}$) in such a way that after this interaction the state will be

$$|\Psi'(0)\rangle = (\sigma_1^+\,e^{ikz_1} + \sigma_1^-\,e^{-ikz_1})\,|\Psi(0)\rangle\,. \tag{21}$$

According to this expression, when ion 1 is in the state $|0\rangle$ it will be transferred to $|1\rangle$, moving at the same time to the right due to the photon absorption. Thus, depending on the internal state of the ion, it will move in two different directions. Due to the Coulomb repulsion, ion 2 undergoes as well a different motion depending on the internal state of ion 1. The distance between these wave packets can be easily calculated by using the Harmonic approximation

$$d(t) = 2z_0\eta\left[\sin\,(\nu_z t) - \frac{1}{2}\,\sin\,(2\nu_z t)\right], \tag{22}$$

where $z_0 = 1/(2m\nu_z)^{1/2}$ ($\hbar = 1$) is the size of the ground state of the ion with the center of mass mode frequency and $\eta = kz_0$ is the corresponding Lamb–Dicke parameter. The maximum distance is given by

$$D \equiv d(t_0) = 3\sqrt{3}\,z_0\eta/2\,, \tag{23}$$

being at $t_0 = 2\pi/(3\nu_z)$.

(ii) Flip $|0\rangle \leftrightarrow |1\rangle$, conditional on ion 2. After a time t_0, a laser beam is applied to ion 2 without inducing any displacement (this can be done using two beams propagating in opposite directions in a Raman transition, or using a laser beam propagating in a direction perpendicular to the axis z for forbidden transitions). The laser is focused on the position $z = (z_e + D)/2$, in such a way that it is interacting only with the wave packet ϕ^R, that is the wave packet that arises if atom 1 was in a state $|0\rangle$. For a given interaction time $t_1 \ll t_g$, this laser induces a rotation $|0\rangle_2 \leftrightarrow |1\rangle_2$. The state after this interaction is given by

$$|\Psi(t_0)\rangle = \left[\sigma_2^x\,e^{-iH_{\text{ho}}t_0}\,\sigma_1^+\,e^{ikz_1} + e^{-iH_{\text{ho}}t_0}\,\sigma_1^-\,e^{-ikz_1}\right]|\Psi(0)\rangle\,, \tag{24}$$

where $\sigma_2^x = \sigma_2^+ + \sigma_2^-$. Later on the ions will oscillate with the free harmonic oscillator Hamiltonian.

(iii) State dependent displacement of ion 1: after a time $t_g - t_0$ we apply a laser pulse to the ion 1 with the wavevector k pointing in the direction of the z axis as in the step (i) for a time $t_{\text{las}} = \pi/\Omega$. Assuming again $t_{\text{las}} \ll t_g$, we can write the state after the interaction as

$$|\Psi(t_g)\rangle = (\sigma_1^+\,e^{ikz_1} + \sigma_1^-\,e^{-ikz_1})\,e^{-iH_{\text{ho}}(t_g - t_0)}\,|\Psi(t_0)\rangle\,. \tag{25}$$

Using eqn. (21) it can be easily seen that

$$|\Psi(t_g)\rangle = \left[\sigma_1^+\sigma_1^- + \sigma_1^-\sigma_1^+\sigma_2^x\right]|\Psi(0)\rangle\,, \tag{26}$$

which coincides with the fundamental logic gate between two qubits defined in (19). According to this equation, it is ideally possible to realize the logical gate without taking into account the state of motion of the ion, pure or mixed. However, the fidelity of the preceding logic gate will be limited in a realistic scenario. In the next section we will discuss some of the main sources of errors causing the departure from the ideal case. Note finally that this scheme is easy to generalize to the case of three ions. Further scaling up is difficult due to the appearance of more frequencies and the difficulty of make them all commensurable

6. Errors in the Computation

We discuss now possible error sources found in the previous schemes. The first type of errors (memory errors) is related to the storage of the quantum register. Errors in the ion register are due to spontaneous decay of the atomic level. This can be avoided by using stable ground state levels. Other effects which lead to memory errors are fluctuations of the trapping fields or imperfect vacuum which causes possible collisions with the background gas. It has been possible to make all these effects very small [8].

The predominant type of errors in both schemes are computational errors. These errors take place only while quantum gates are performed and may have a variety of causes. One type of error is decoherence during quantum gates. This error will be relevant if the qubits involved in the gate are coupled to fragile auxiliary systems, e.g. the CM mode acting as a bus of quantum information in the cold scheme. Other types of errors occur when the transformations needed to perform the quantum gate are not accurate, e.g. this could be the case when the wave packets overlap in the hot gate.

In the *cold* gate scheme quantum gates require the excitation of the CM phonon mode temporarily. This is a much more fragile quantum system than the internal (electronic) states. The CM motion undergoes damping and heating due to its coupling to the electrodes of the trap. Moreover, as we discussed previously, errors in the CM motion can also arise due to imperfect laser cooling, and this is the main motivation for the introduction of the hot scheme. It is clear that a single error in the phonon mode can damage the quantum state which is involved in the gate and, by error propagation the whole quantum computation.

Although general error correction methods have been developed which can cope with computational errors [10], these methods require an extensive computational overhead and may not be applicable at all to prototype realizations like the ion trap quantum computer. Recently, a scheme to correct errors in the CM phonon mode has been introduced [11]. The goal was to perform the conditional sign–change gate, eqn. 8, in a faul–tolerant way. The first step of the scheme is to encode the two qubits involved in the gate redundantely in four qubits. This is done in such way that each qubit is encoded in two qubits situated on the same ion (the logical qubit is distributed onto four electronic levels). In particular, the logical values 0 and 1 are encoded into two physical qubits (on a single ion) as follows

$$|0\rangle_L = (|00\rangle + |11\rangle)/\sqrt{2}, \qquad |1\rangle_L = (|01\rangle + |10\rangle)/\sqrt{2}\,. \tag{27}$$

This is made by single ion operations which are considered error–free. The gate is performed on the codeword as a sequence of four subgates between physical qubits and with the difference that we tune the lasers such that the ground state $|0\rangle_{\mathrm{CM}}$ is coupled directly to the second excited state $|2\rangle_{\mathrm{CM}}$, instead of the $|1\rangle_{\mathrm{CM}}$ state. If a decay takes place, a superposition of these states undergoes the transformation $\alpha|0\rangle_{\mathrm{CM}} + \beta\,|2\rangle_{\mathrm{CM}} \to |1\rangle_{\mathrm{CM}}$. After each subgate an error measurement is realized. Thus if any population is found in the phonon state $|1\rangle_{\mathrm{CM}}$ we have detected an error. One can show that the two physical qubits not involved in the particular subgate carry enough information to reconstruct the state before the subgate. After the reconstruction the subgate can be performed again. After all four subgates have been performed succesfully we can reverse the initial coding step in order to decode the redundant enconding.

Let us consider now the hot scheme. In this case, the main sources of errors in the realization of the gate will be related to two factors:

(a) The finite size of the wave packets and the small distances between them. In particular this can cause the following problems: (a.1) $\phi_2^{R,L}$ might overlap in the time t_0, preventing us from addressing only one of the wave packets; (a.2) even if there is no overlapping between the wave packets, it would be almost impossible to focus a laser beam in such small distances; (a.3) due to the laser spatial profile, different positions within the laser beam will see different laser intensities.

Let us consider the relevant parameters related to this problem. Associated to the wave packets of ion 2 at time t_0 we have two parameters Δ and d corresponding respectively to the size of the wave packets and their separation, Eqn. (28). In order to overcome problem (a.1) it is required that $D \gg \Delta$. Other important parameters are those of the laser profile. This is characterized by the position dependent Rabi frequency $\Omega(x)$ which takes on a maximum value Ω_0 and has a width W. In present experiments, due to the impossibility of focusing laser beams over small distances, this width is expected to be much larger than the separation. Thus

$$W \gg D \gg \Delta\,. \tag{28}$$

Consequently, the laser beam will affect both wave packets $\varphi^{R,L}$, which causes problem (a.2)

The solution to this problem is to select the laser parameters so that $\phi_2^{R,L}$ feel different Rabi frequencies, $\Omega^{R,L}$, fulfilling

$$\frac{\Omega^R}{2}\,t_1 = (2N + 1/2)\,\pi\,, \qquad \frac{\Omega^L}{2}\,t_1 = 2N\pi\,, \tag{29}$$

with intenger N equal to the number of complete Rabi cycles. On the other hand, we have to make sure that the whole wavepacket sees basically the same Rabi frequency, so that no information of the internal state is imprinted in the motional state; i.e. we have to overcome problem (a.3). According to (28) this requires that

$$\left.\frac{d\Omega(z)}{dx}\right|_{z=\bar{z}_2^{R,L}} \Delta \ll 1, \tag{30}$$

where $z = \bar{z}_2^{R,L}$ is the position of the center of the wave packets of atom 2 when interacting with the laser. In order to illustrate the above conditions, we consider the case of a thermal state, characterized by $\bar{n}_c$ and $\bar{n}_r = \bar{n}_c^2/(2\bar{n}_c + 1)$, the mean phonon number of the center–of–mass and relative modes. For the laser profile we take a Gaussian $\Omega(z) = \Omega_0 \exp\left[-(z-l)^2/(2W^2)\right]$. We choose the equilibrium point of atom 2 to coincide with the steepest point of the laser profile, i.e. $l = z_e/2 + W$. The condition (28) can now be expressed as $W \gg 3\sqrt{3}\,z_0\eta/2 \gg \sqrt{\bar{n}_c + \bar{n}_r/2 + 3/4}\,z_0$ taking into account (23). According to the first inequality, we can expand the Gaussian profile $\Omega(x)$ around $z = z_e/2$ up to first order. Imposing now the second condition (29) we obtain

$$\frac{\Omega(z_e/2)}{2}\, t_1 = (2N + 1/4)\, \pi\,, \tag{31}$$

and $W = (4N + 1/2)\, D$. Having this in mind, the first condition (28) can be restated as

$$4N \gg 1\,, \qquad \eta \gg \sqrt{4\bar{n}_c + 2\bar{n}_r + 3}/(3\,\sqrt{3})\,. \tag{32}$$

With this, condition (30) is automatically fulfilled. In summary, the laser parameters have to be chosen following the conditions (31) and (32).

(b) The nonharmonic corrections to the free Hamiltonian. The fact that the wave packets would separate considerably, since this seems to be a solution to the above mentioned problems, would imply that the nonharmonic terms could become important. In order to single out these effects, we will assume that the action of the laser beam in step (ii) is ideal. In that case, the errors caused by anharmonicities are independent of the internal dynamics, which simplifies the analysis. The fidelity of the gate will then be simply given by the overlap $\mathcal{F}_{\rm cor} = |\langle \Psi(t_g)\, |\Psi(0)\rangle|^2$. As it should be, if we set $V_{\rm cor} = 0$ we will have $\mathcal{F}_{\rm cor} = 1$ according to (15). Using time-dependent perturbation theory we find $\mathcal{F}_{\rm cor} = 1 - (\Delta\tilde{V})^2$, where

$$\tilde{V} = \int_0^{t_g} d\tau\, e^{iH_{\rm ho}\tau}\, V_{\rm cor}\, e^{-iH_{\rm ho}\tau}. \tag{33}$$

and $(\Delta\tilde{V})^2 = \langle\Psi(0)|\, \tilde{V}^2\, |\Psi(0)\rangle - \langle\Psi(0)|\, \tilde{V}\, |\Psi(0)\rangle^2$. This fidelity can then be evaluated analytically for an initial thermal state. Taking into account realistic parameters it seems that the errors produced by the anharmonicities can be neglected with respect to the ones due to the finite size effect of the wave packets for small values of the Lamb–Dicke parameter.

7. Conclusions and Outlook

In this paper we have reviewed two complementary schemes for quantum computation with trapped ions. We compare now briefly both proposals by listing some of the main features and differences between them

1. $1_{\rm cold}$) Combined laser cooling techniques are needed. In particular, Doppler cooling to the Doppler cooling limit followed by sideband cooling to the ground motional state. $1_{\rm hot}$) Doppler cooling is the only cooling technique needed.
2. $2_{\rm cold}$) It is necessary to work in the Lamb–Dicke limit, both for laser cooling and for manipulation of the ions. $2_{\rm hot}$)The Lamb–Dicke limit is not needed.
3. $3_{\rm cold}$) Low intensity laser beams, i.e. $\Omega\eta \ll \nu$, the so-called weak excitation limit (where here Ω is the Rabi frequency, η denotes the Lamb–Dicke parameter and ν the frequency of the trap). This implies a clock time for the *cold* gate of $\tau_{\rm cold} \approx (\eta\Omega)^{-1} \gg \nu^{-1}$. $3_{\rm hot}$) The clock time associated to the logic gate is faster, being typically characterized by the trap oscillation time, i.e. $\tau_{\rm hot} \approx \nu^{-1} \ll \tau_{\rm cold}$
4. $4_{\rm cold}$) Scalability is possible, although technologically demanding. Small processors with 5–10 ions seem to be possible in the near future. $4_{\rm hot}$) Scalability to more than three ions is not possible. On the other hand, the technology associated to the realization of the logic gate presents fewer technical problems.

5. 5_{cold}) Ions are manipulated in a regime similar to quantum electrodynamics and the physics of generating and manipulating N-particles entangled states in ion traps and high−Q cavities, i.e. $\nu \gg \Omega$, see [12]. 5_{hot}) Ions are manipulated in a different regime, similar to atom interferometry, i.e. $\nu \ll \Omega$, see [13].

The schemes presented in this paper are only a starting point to acquire first experimental evidences of some of the interesting effects associated with quantum computing. Given the experimental progress in several laboratories worldwide, we expect that model systems of ion trap quantum computers with five to ten qubits will be built within the coming years. These systems will provide the context for a new generation of fundamental tests of quantum mechanics, and for the demonstration of basic elements of particle entanglement, quantum computing, error correction, and quantum communication [14].

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