Quantum Gates with "Hot" Trapped Ions

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We propose a scheme to perform a fundamental two-qubit gate between two trapped ions using ideas from atom interferometry. As opposed to the scheme considered by J. I. Cirac and P. Zoller, [Phys. Rev. Lett. **74**, 4091 (1995)], it does not require laser cooling to the motional ground state. [S0031-9007(98)06818-5]

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Quantum computation allows the development of polynomial-time algorithms for computational problems such as prime factoring, which have previously been viewed as intractable [1]. This has motivated studies into the feasibility of actual implementation of quantum computers in physical systems [2]. The task of designing a quantum computation is equivalent to finding a physical realization of quantum gates between a set of qubits, where a qubit refers to a two-level system $\{|0\rangle, |1\rangle\}$. Any operation can be decomposed into rotations on a single qubit and a fundamental two-bit gate [3], such as, for example, the one in which the second (target) qubit is flipped when the first (control) is in $|0\rangle$, i.e.,

$$\hat{C}_{12} : |\epsilon_1\rangle|\epsilon_2\rangle \to |\epsilon_1\rangle|(1 - \epsilon_1) \oplus \epsilon_2\rangle \qquad (\epsilon_{1,2} = 0, 1), \tag{1}$$

where $\oplus$ denotes addition mod 2. Achieving the conditions for quantum computation is extremely demanding, and only a few systems have been identified as possible candidates to build small scale models in the lab [1,2,4,5]. One of the most promising examples is a string of cold ions stored in a linear trap [4]. In this case qubits are stored in long-lived internal atomic ground states. Single bit operations are accomplished by directing laser beams to each of the ions; a fundamental gate is implemented by exciting the collective quantized motion of the ions with lasers, where the exchange of phonons serves as a data bus to transfer quantum information between the qubits. In this scheme ground state cooling [6] is required. Otherwise, the phonons introduced during the gate will be indistinguishable from those already existing at finite temperature, leading to errors even for $\overline{n} \gtrsim 1$, where $\overline{n}$ denotes the mean phonon number in the fundamental mode. While ground state cooling can be achieved for high trap frequencies (of the order of several MHz [7]) new laser cooling methods need to be developed for smaller trap frequencies (of the order of 100 kHZ) where most of the linear ion trap experiments operate. For those experiments, it would be of fundamental relevance to develop methods to perform gates without the zero temperature requirement.

In this Letter we discuss the implementation of a fundamental two-bit gate between two ions in a linear trap at finite temperature. Our proposal is fundamentally different to the one of Ref. [4]: Here [Fig. 1(a)], we split the wave packet of the control ion (ion 1) into two (ϕ_1^R and ϕ_1^L) that move along different directions (right and left) depending on its internal state. The Coulomb interaction produces a splitting of the wave packet of the target ion (ion 2) into two (ϕ_2^R and ϕ_2^L) correspondingly. Then a laser is focused to the center of the wave packet ϕ_2^R, producing a spin flip. Finally, the first ion is returned to its original motional state giving rise to (1). The novel concept behind the gate operation is the splitting of the atomic wave packets in the same way as one does in atom interferometry [8,9] and then performing conditional dynamics with respect to the spatial location of the wave packets. Under appropriate conditions the performance of the gate is independent of the motional state of the ions and therefore zero temperature is not required. On the other hand, the scheme requires for the motion of the ions to be periodic. Since for an external harmonic trap the motion is not strictly periodic, this implies that an anharmonic trap has to be used. Linear ion traps can be easily made anharmonic with the addition of external static fields or by changing the charge distribution of the electrodes.

In order to analyze the scheme, we consider two ions confined in a linear trap [10]. We will assume that the motion of the ions is frozen with respect to the y and z

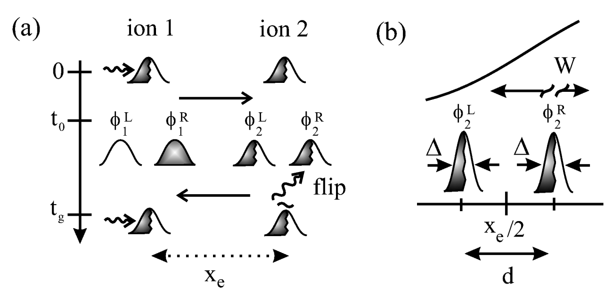

FIG. 1. (a) Lasers and ions configuration in the trap (see text). (b) Detailed configuration for the conditional laser beam on ion 2. The dark (bright) fill denotes an excited (ground) internal state. The tilde above the wave packet indicates the conditional spin flip in ion 2.

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axes. The Hamiltonian describing this situation is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2) + \frac{e^2}{4\pi\epsilon_0|x_2 - x_1|}, \quad (2)$$

where V denotes the external confining potential along the x axis, which we assume to be symmetric $V(x) = V(-x)$. We denote by x_e the separation of the ions at the equilibrium position. Using the center-of-mass and the relative coordinates,

$$x_c = (x_1 + x_2)/2, \quad x_r = x_2 - x_1 - x_e, \quad (3a)$$

$$p_c = p_1 + p_2, \quad p_r = (p_2 - p_1)/2, \quad (3b)$$

and expanding around the equilibrium position $x_c = x_r = 0$ up to second order, we obtain $H = H_{\rm ho} + V_{\rm cor}(x_c, x_r)$, where

$$H_{\rm ho} = \frac{p_c^2}{2m_c} + \frac{p_r^2}{2m_r} + \frac{1}{2} m_c \nu_c^2 x_c^2 + \frac{1}{2} m_r \nu_r^2 x_r^2, \quad (4)$$

and $V_{\rm cor}$ denotes the third and higher-order corrections. Here $m_c = 2m$, $m_r = m/2$, and $\nu_{c,r}$ are the corresponding frequencies in the harmonic approximation

$$\nu_c^2 = \frac{1}{m_c} \frac{\partial^2 V}{\partial x_c^2}\bigg|_{eq}, \qquad \nu_r^2 = \nu_c^2 + \frac{e^2}{\pi\epsilon_0 m x_e^3}. \quad (5)$$

In the following we will assume that the trapping potential is such that ν_r and ν_c are commensurable, i.e., the motion is strictly periodic with period t_g. In practice this can be done by adding static external fields so that the potential $V(x)$ becomes anharmonic. One can then change the fields continuously until one reaches the point where the frequencies are commensurable, which can easily be measured. In that case, and provided $\overline{V}_{\rm cor} t_g \ll 1$, where $\overline{V}_{\rm cor}$ denotes the typical values of the corrections,

$$e^{-iH_{\rm ho}t_g} = 1. \quad (6)$$

In the following, and for the sake of simplicity, we will assume $\nu_r = 2\nu_c$.

We will denote by $|0\rangle$ and $|1\rangle$ the two internal states of atoms which store the quantum information, and by $|\Psi(0)\rangle = |\Psi_{\rm int}\rangle \otimes |\Psi_{\rm mot}\rangle$ the initial state of the ions. Although we have taken both pure internal $|\Psi_{\rm int}\rangle$ and motional states $|\Psi_{\rm mot}\rangle$, our analysis is also valid for mixed initial states. The gate is performed in three steps:

(i) State dependent kick on atom 1.—A short laser pulse is applied to atom 1 with a k vector pointing along the x direction [11]. The laser intensity is chosen such that the internal state of the atom undergoes a flip. The motional state of the ions is also changed by this laser beam. We write the Hamiltonian for the atom-laser interaction as $H_{\rm las} = \frac{\Omega}{2}(\sigma_1^+ e^{ikx_1} + \sigma_1^- e^{-ikx_1})$, where Ω is the Rabi frequency, $\sigma^+ = |1\rangle\langle 0| = (\sigma^-)^\dagger$, and the subscript denotes from now on which atom is addressed. The laser is applied for a time $t_{\rm las} = \pi/\Omega \ll t_g$ (the so-called strong-excitation regime, see [8] for details), so that the state after this interaction is

$$|\Psi'(0)\rangle = (\sigma_1^+ e^{ikx_1} + \sigma_1^- e^{-ikx_1})|\Psi(0)\rangle. \quad (7)$$

According to this expression, if atom 1 is in the state $|0\rangle$ it will be transferred to the state $|1\rangle$ and undergo a photon kick which will push it to the right. If it is initially in $|1\rangle$ it will be transferred to the state $|0\rangle$ and undergo a photon kick towards the left [see Fig. 1(a)]. Thus, depending on the internal state of the atom it will start moving in different directions. Because of the Coulomb repulsion, atom 2 will also undergo a different motion depending on the initial state of atom 1. That is, the motional state of the second atom will also split into two wave packets, ϕ_2^R and ϕ_2^L, moving in different directions (right and left). One can easily calculate the evolution of the distance between the center of these wave packets using the harmonic Hamiltonian (4)

$$d(t) = 2x_0\eta\left[\sin(\nu_c t) - \frac{1}{2}\sin(2\nu_c t)\right], \quad (8)$$

where $x_0 = 1/(2m\nu_c)^{1/2}$ ($\hbar = 1$) is the ground state size of a single ion with the center-of-mass mode frequency and $\eta = kx_0$ the corresponding Lamb-Dicke parameter. The maximum distance is

$$D \equiv d(t_0) = 3\sqrt{3}\,x_0\eta/2, \quad (9)$$

and is produced at $t_0 = 2\pi/(3\nu_c)$.

(ii) Conditional flip on atom 2.—After a time t_0, a laser beam is applied to atom 2, such that it does not suffer a kick [12]. The laser is focused on the position $x = (x_e + D)/2$, so that it overlaps only with the wave packet ϕ_2^R, that is, the one that arises if atom 1 was in state $|0\rangle$. Adjusting properly the interaction time $t_1 \ll t_g$, it induces a rotation $|0\rangle_2 \leftrightarrow |1\rangle_2$. The state after this laser interaction will be

$$|\Psi(t_0)\rangle = [\sigma_2^x e^{-iH_{\rm ho}t_0}\sigma_1^+ e^{ikx_1} + e^{-iH_{\rm ho}t_0}\sigma_1^- e^{-ikx_1}] \times |\Psi(0)\rangle, \quad (10)$$

where $\sigma_2^x = \sigma_2^+ + \sigma_2^-$. After this interaction, the atoms continue their evolution with the free harmonic oscillator Hamiltonian.

(iii) State dependent kick of atom 1.—After a time $t_g - t_0$ a short laser pulse is applied to atom 1 with a k vector pointing along the x direction as in the step (i) for a time $t_{\rm las} = \pi/\Omega$. Assuming again $t_{\rm las} \ll t_g$, we can write the state after this interaction as

$$|\Psi(t_g)\rangle = (\sigma_1^+ e^{ikx_1} + \sigma_1^- e^{-ikx_1})e^{-iH_{\rm ho}(t_g - t_0)}|\Psi(t_0)\rangle. \quad (11)$$

Using (6) one can easily check that

$$|\Psi(t_g)\rangle = [\sigma_1^+\sigma_1^- + \sigma_1^-\sigma_1^+\sigma_2^x]|\Psi(0)\rangle, \quad (12)$$

which coincides with the fundamental two-bit quantum gate (1).

Under ideal conditions the above steps allow one to perform a two-bit quantum gate. According to (12) the operators acting on the motional state cancel out, and therefore the gate can be carried out independent of the motional state, regardless of whether it is pure or mixed. In a real situation, there will be restrictions to accomplish the gate with high fidelity. The main sources of errors will be related to (a) the finite size of the wave packets and their small separations and (b) the nonharmonic corrections to the free Hamiltonian (4).

The finite size effects can cause several problems during step (ii): (1) $\phi_2^{R,L}$ may overlap at time t_0, which will prevent us from addressing one of them alone with the laser beam; (2) even if the wave packets do not overlap, it will be hard (if not impossible) to focus a laser beam to such small distances; (3) due to the spatial profile of the laser beam different positions within the wave packet will see different laser intensities. On the other hand, if the wave packets separate from each other considerably, since this seems to be one way to avoid some of the above-mentioned problems, the anharmonic terms due to V_{cor} may become important. In the following, we will address all of these questions, find the conditions under which these problems can be overcome, and present numerical simulations showing the performance of the scheme in realistic setups.

Let us consider first the problems related to the finite size of the atomic wave packets and their small separations in step (ii). We denote by Δ the size of the wave packets corresponding to atom 2 at time t_0 and by d their separation (8) [Fig. 1(b)]. In order to overcome problem (1) it is required $D \gg \Delta$. The laser profile is characterized by the position dependent Rabi frequency $\Omega(x)$ which takes on a maximum value Ω_0 and has a width W. In present experiments, due to the impossibility of focusing laser beams over small distances, this width is expected to be much larger than the separation. Thus,

$$W \gg D \gg \Delta \,. \tag{13}$$

Consequently, the laser beam will affect both wave packets $\phi_2^{R,L}$, which causes problem (2). The solution to this problem is to select the laser parameters so that $\phi_2^{R,L}$ feel different Rabi frequencies, $\Omega^{R,L}$, fulfilling

$$\frac{\Omega^R}{2}\, t_1 = (2N + 1/2)\pi, \qquad \frac{\Omega^L}{2}\, t_1 = 2N\pi \,, \tag{14}$$

with N integer, the number of complete Rabi cycles. On the other hand, we have to make sure that the whole wave packet sees basically the same Rabi frequency, so that no information of the internal state is imprinted in the motional state; i.e., we have to overcome problem (3). According to (13) this requires that

$$\left.\frac{d\Omega(x)}{dx}\right|_{x=\overline{x}_2^{R,L}} \Delta \ll 1 \,, \tag{15}$$

where $x = \overline{x}_2^{R,L}$ is the position of the center of the wave packets of atom 2 when interacting with the laser. In order to illustrate the above conditions, let us consider that the ions are initially in a thermal state, characterized by $\overline{n}_c$, the mean phonon number in the center-of-mass mode, and by $\overline{n}_r = \overline{n}_c^2/(2\overline{n}_c + 1)$, the one in the relative motion. For the laser profile we take a Gaussian $\Omega(x) = \Omega_0 \exp[-(x - l)^2/(2W^2)]$. We choose the equilibrium point of atom 2 to coincide with the steepest point of the laser profile, i.e., $l = x_e/2 + W$. The condition (13) can now be expressed as $W \gg 3\sqrt{3}\, x_0 \eta/2 \gg \sqrt{\overline{n}_c + \overline{n}_r/2 + 3/4}\, x_0$ taking into account (9). According to the first inequality, we can expand the Gaussian profile $\Omega(x)$ around $x = x_e/2$ up to first order. Imposing now the second condition (14) we obtain

$$\frac{\Omega(x_e/2)}{2}\, t_1 = (2N + 1/4)\pi \,, \tag{16}$$

and $W = (4N + 1/2)D$. Having this in mind, the first condition (13) can be restated as

$$4N \gg 1, \qquad \eta \gg \sqrt{4\overline{n}_c + 2\overline{n}_r + 3}/(3\sqrt{3}) \,. \tag{17}$$

With this, condition (15) is automatically fulfilled. In summary, the laser parameters have to be chosen following the conditions (16) and (17).

In order to illustrate to what degree the conditions derived above have to be fulfilled, we have performed some numerical calculations. We have considered as an example the potential $V(x) = K|x|^{5/3}$, which ensures that $\nu_r = 2\nu_c$. We emphasize that any potential [for example, of the form $V(x) = ax^2 + bx^4$ [13]] which produces commensurable frequencies will give the same result [14]. We have constructed the evolution according to the harmonic approximation, and calculated the averaged fidelity $\mathcal{F}$ and purity $\mathcal{P}$ (Ref. [15]) for different temperatures, i.e., the mean center-of-mass phonon number $\overline{n}_c$, and values of η. These quantities characterize the performance of the gate, and the degree of decoherence, respectively. In Figs. 2(a) and 2(b) one can clearly see that to obtain a good fidelity for high temperatures ($\overline{n}_c$) one has to increase η [see Eq. (17)]. We emphasize that if in a given experiment η is not large enough, one can simply apply a sequence of π pulses, from the left and right alternatively, to increase the effective displacement of the wave packets [16]. For example, the $S_{1/2} \to D_{5/2}$ transition of Ca^+ has $\eta \approx 0.45$, for a trap frequency $\nu_c = 2\pi \times 50$ kHz. In order to obtain an effective value of $\eta = 7$ one should apply a sequence of the order of 15 pulses. On the other hand, the maximum separation $D \approx 5\ \mu$m. Note that this corresponds to typical separation distances in atom interferometry [9].

We consider now the second source of errors, namely, the effects of the anharmonic term V_{cor} that we have neglected so far. In order to single out these effects, we will assume that the action of the laser beam in step (ii) is ideal. In that case, the errors caused by anharmonicities are

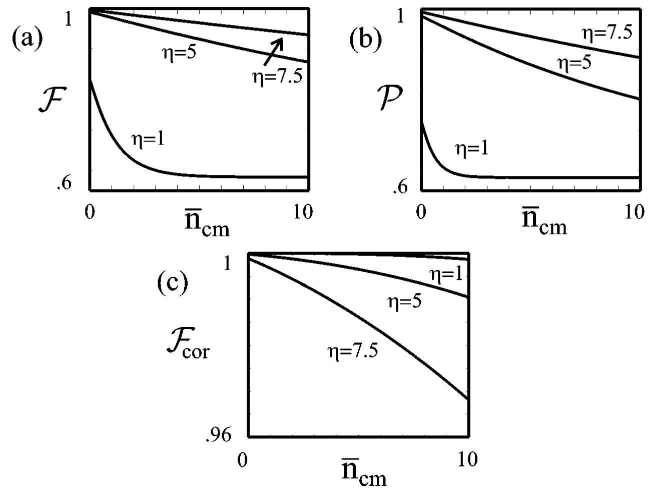

FIG. 2. (a) Fidelity and (b) purity as a function of the mean phonon number of the center-of-mass mode. (c) Fidelity with respect to the anharmonicities as a function of the mean phonon number of the center-of-mass mode. The values of the Lamb-Dicke parameter are indicated. The rest of the parameters correspond to those of the Ca^+ ion.

independent of the internal dynamics, which simplifies the analysis. The fidelity of the gate will be then simply given by the overlap $\mathcal{F}_{\rm cor} = |\langle\Psi(t_g)|\Psi(0)\rangle|^2$. As it should be, if we set $V_{\rm cor} = 0$ we will have $\mathcal{F}_{\rm cor} = 1$ according to (6). Using time-dependent perturbation theory we find $\mathcal{F}_{\rm cor} = 1 - (\Delta\tilde{V})^2$, where

$$\tilde{V} = \int_0^{t_g} d\tau e^{iH_{\rm ho}\tau} V_{\rm cor} e^{-iH_{\rm ho}\tau}, \tag{18}$$

and $(\Delta\tilde{V})^2 = \langle\Psi(0)|\tilde{V}^2|\Psi(0)\rangle - \langle\Psi(0)|\tilde{V}|\Psi(0)\rangle^2$. This fidelity can be evaluated analytically for an initial thermal state. In Fig. 2(c) we have plotted $\mathcal{F}_{\rm cor}$ as a function of η and $\bar{n}_c$. Comparing with Figs. 2(a) and 2(b), we see that the errors produced by anharmonicities can be neglected with respect to the ones due to the finite size effects of the wave packets, at least for the values represented in these plots. For larger values of η, however, the anharmonicity must be taken into account.

We have not considered the effects of decoherence during the gate operations. The fundamental limits, however, will allow one to perform many gate operations during the decoherence time [10,17].

In summary, we have shown how to implement two-bit quantum gates between two ions in a linear trap at nonzero temperature. The scheme can be easily generalized to the case of three ions. Further scaling up is not easy since one should tune the trap potentials so that the different frequencies involved become commensurable. We expect the present proposal to be of interest in application of quantum logic with two and three qubits, e.g., fundamental experiments involving particle entanglement, error correction schemes [1], and quantum communication [18,19].

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[11] By using two focused Raman beams propagating such that the different of the k vectors lies along the trap axis.
[12] Using two copropagating beams in a Raman transition, or using a laser beam propagating in a direction perpendicular to the axis x for forbidden transitions.
[13] To have $\nu_r/\nu_c = 1.5$ one needs $a/b = 7x_e^2/6$. Taking the configuration of H. C. Nägerl *et al.* [Appl. Phys. B **66**, 603 (1998)] with two rings of radius $R = 2$ mm separated by L, we get $L/R \simeq \sqrt{2}$ (which implies $L \simeq 3$ mm). The corresponding potential would be $\sim 10^4$ V.
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